Indian Statistical Institute, Bangalore
B. Math.
First Year, Second Semester
Real Analysis-II

Final Examination | May 2, 2024
Maximum marks: 100 | Time: 3 hours
Instructor: B V Rajarama Bhat

In the following a, b are real numbers with $a < b$.

(1) Suppose $f : [a, b] \to \mathbb{R}$ is a continuous function. Show that f is Riemann integrable. [15]

(2) Suppose $f_0 : [0, 1] \to \mathbb{R}$ is a Riemann integrable function. Define a sequence of functions, $\{f_n\}_{n\geq 1}$ inductively, by setting:

$$f_n(x) = \int_0^x f_{n-1}(t)dt, \ n \geq 1$$

Suppose $|f_0(t)| \leq M$ for every $t \in [0, 1]$, show that for $n \geq 0$,

$$|f_n(x)| \leq M\frac{x^n}{n!}, \ 0 \leq x \leq 1.$$

Show that $\{f_n\}_{n\geq 0}$ converges to 0 uniformly. [15]

(3) Let N be the space of real valued continuous functions on $(0, 1]$. Define N_0 by

$$N_0 = \{f \in N : \lim_{x\to 0} f(x) = 0\}$$

(i) Show that every f in N_0 is bounded. (ii) If $\{f_n\}_{n\geq 1}$ is a sequence of functions in N_0 converging uniformly to a function f in N. Show that $f \in N_0$. [15]

(4) Let $\{q_1, q_2, \ldots, \}$ be an enumeration of rational numbers of $[0, 1]$. For $n \geq 1$, define $g_n : [0, 1] \to \mathbb{R}$ by

$$g_n(x) = \begin{cases} q_j & \text{if } x = q_j, \text{ for some } j, \text{ where } 1 \leq j \leq n; \\ 0 & \text{otherwise.} \end{cases}$$

(i) Show that $\{g_n\}_{n\geq 1}$ converges pointwise to a limit function g. (ii) Show that the convergence in (i) is not uniform. (iii) Show that every $\{g_n\}_{n\geq 1}$ is Riemann integrable, but the limit function g is not Riemann integrable. [15]

(5) Suppose $\{f_n\}_{n\geq 1}$ is a sequence of continuous functions on $[a, b]$ converging uniformly to a function f on $[a, b]$. Show that f is continuous. [15]

(6) Show that

$$\lim_{n\to\infty} \frac{1}{8^n} \sum_{k=0}^{n} \sqrt[3]{\frac{k}{n}} \begin{pmatrix} n \\ k \end{pmatrix} 7^k = \frac{\sqrt[3]{7}}{2}.$$

(Hint: Use Bernstein polynomial approximation of appropriate function.). [15]

(7) Obtain a power series expansion for the function

$$h(x) = \frac{1}{x-2} + \frac{1}{x+5}, \ x \in [-1, 1].$$

Justify your claim. [15]